Tamm-Dancoff as the contraction limit of the Richardson-Gaudin equations for pairing

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1 Pairing in the Shell Model

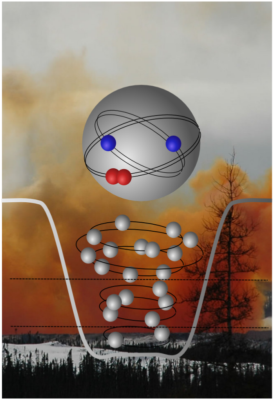
2 An integrable model

- Richardson-Gaudin
- Tamm-Dancoff Approximation
- From TDA to RG

3 Summary

- Remarks and Conclusions

The shell model



- The shell model assumes that the nucleons *approximately* live as independent particles within a mean field, generated by the others.
- Not all interactions are in the mean field. *n*-body *residual* interactions can be taken into account

$$\hat{H} = \sum_{\alpha} \varepsilon_\alpha a^\dagger_\alpha a_\alpha + \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \gamma\delta \rangle a^\dagger_\alpha a^\dagger_\beta a_\gamma a_\delta$$

- This is the shell-model Hamiltonian

Pairing in the shell model

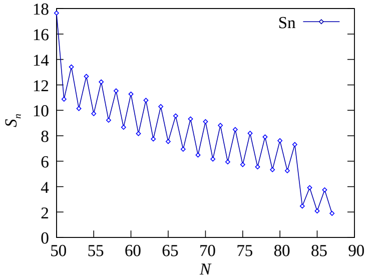
The interaction can be developed in a total angular momentum J expansion

$$\hat{H} = \sum_a \varepsilon_a \hat{n}_a + \frac{1}{4} \sum_J \sum_{abcd} \langle ab, JM|V|cd, JM\rangle [a^\dagger_{j_a} a^\dagger_{j_b}]^{(J)} \cdot [\tilde{a}_{j_c} \tilde{a}_{j_d}]^{(J)}$$

For short-range interactions, the $J = 0$ components are predominant.

The pairing Hamiltonian

$$\hat{H} = \sum_a \varepsilon_a \hat{n}_a + \tfrac{1}{4} g \sum_{ac} (a^\dagger_{j_a} \cdot a^\dagger_{j_a})(\tilde{a}_{j_c} \cdot \tilde{a}_{j_c})$$



Algebraic properties

quasi spin

The *quasi-spin* algebra is spanned by

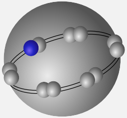
$$su(2)_j \left\{ \begin{array}{l} \hat{S}^0_j = \frac{1}{2}\hat{n}_j - \frac{1}{4}\Omega_j \\ \hat{S}^\dagger_j = \frac{1}{2}(a^\dagger_j \cdot a^\dagger_j) \\ \hat{S}_j = (\hat{S}^\dagger_j)^\dagger \end{array} \right.$$

The corresponding *irreps* are

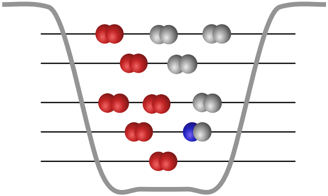
$$|S_j, M_{S_j}\rangle = |\tfrac{1}{4}\Omega_j - \tfrac{1}{2}v_j; \tfrac{1}{2}n_j - \tfrac{1}{4}\Omega\rangle$$

- Ω_j : degeneracy
- v_j : seniority
- pair creation

$$(\hat{S}^\dagger_{\frac{11}{2}})^0|\theta\rangle = |\tfrac{5}{2}, -\tfrac{5}{2}\rangle$$



- k-dim pairing problem has an $su(2)_{j_1} \oplus su(2)_{j_2} \oplus \cdots \oplus su(2)_{j_k}$ spectrum generating algebra



- Dimensions easily become huge
- BCS : non-canonical approximation, finite-size effects

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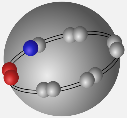

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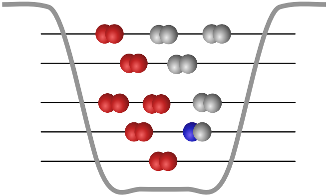

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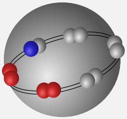

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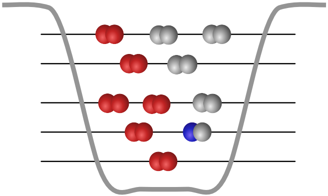

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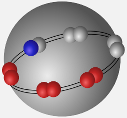

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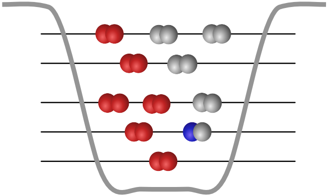

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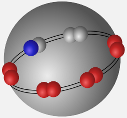

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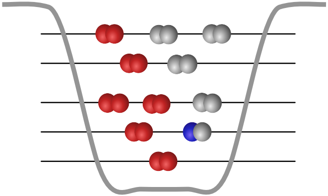

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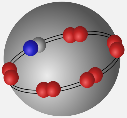

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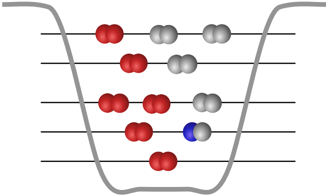

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An exactly solvable model

- The pairing problem is known to be an exactly solvable model
- The Hamiltonian can be diagonalised using a Bethe Ansatz wavefunction

$$|\psi\rangle = \prod_{\alpha=1}^{N} S_\alpha^\dagger |\theta\rangle \qquad \text{with} \qquad S_\alpha^\dagger = \sum_i \frac{S_i^\dagger}{2\varepsilon_i - E_\alpha}$$

- provided the parameters E_α fullfill the

Richardson-Gaudin (RG) equations

$$1 - 2g \sum_{i=1}^{k} \frac{\frac{1}{2}v_i - \frac{1}{4}\Omega_i}{2\varepsilon_i - E_\alpha} - 2g \sum_{\beta \neq \alpha}^{N} \frac{1}{E_\beta - E_\alpha} = 0 \qquad (\forall \alpha = 1 \dots N)$$

- with the eigenstate energy given by

$$E = \sum_{\alpha=1}^{N} E_\alpha + \sum_{i=1}^{k} \varepsilon_i v_i .$$

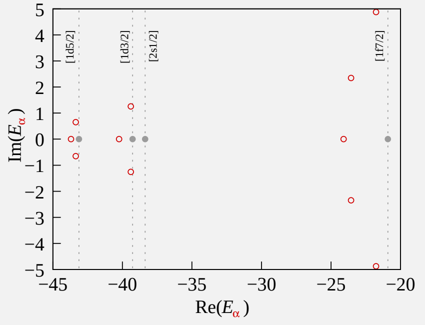
The RG variables

Example: neutron superfluidity in ^{56}Fe. 11 pairs in 10 Wood-Saxon levels ($g = -0.5$)

Level (i)	(Ω_i)	Energy (ε_i)
$1d_{5/2}$	6	-21.5607
$1d_{3/2}$	4	-19.6359
$2s_{1/2}$	2	-19.1840
$1f_{7/2}$	8	-10.4576
$2p_{3/2}$	4	-8.4804
$1f_{5/2}$	6	-7.7003
$2p_{1/2}$	2	-7.6512
$3s_{1/2}$	2	-0.3861
$2d_{5/2}$	6	0.2225
$1g_{9/2}$	10	0.5631

- The RG variables are complex
- They come in cc^* pairs
- Organised around s.p. levels

distribution in the Re-Im plane



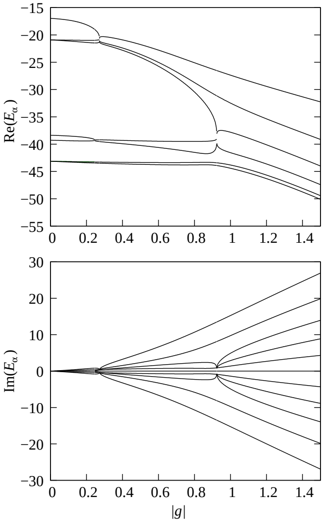
The RG variables: critical points

- The RG variables are highly coupled

$$1 - 2g\sum_{i=1}^{k}\frac{\frac{1}{2}v_i - \frac{1}{4}\Omega_i}{2\varepsilon_i - E_\alpha} - 2g\sum_{\beta\neq\alpha}^{N}\frac{1}{E_\beta - E_\alpha} = 0$$

- Every ε_i is a static attractive pole for E_α
- Every E_β is a dynamic repulsive pole for E_α
- This gives rise to singularities around so-called critical interaction strengths
- Cluster method [1] solves adiabatically through the singularities

Rombouts S, Van Neck D and Dukelsky J 2004 *Phys. Rev. C* **69** 061303(R)



Tamm-Dancoff Approximation (RPA)

- Assume the wavefunction has a bosonic character (approximately)

$$\hat{H}(b^\dagger)^n|\theta\rangle = E_n(b^\dagger)^n|\theta\rangle$$

- $b^\dagger (= \sum_i^k c_i S_i^\dagger)$ is the boson creator
- The 1-boson eigenvalue equation

$$\hat{H}b^\dagger|\theta\rangle = E_1 b^\dagger|\theta\rangle$$

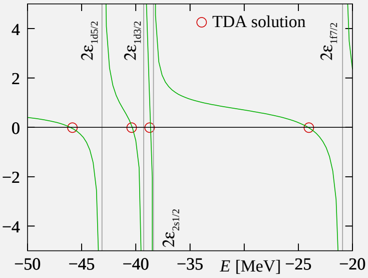
- We can construct the n-boson state from the 1-boson operators

$$|n_1, n_2, \ldots, n_k\rangle = \prod_i \frac{1}{\sqrt{n_i!}}(b_i^\dagger)^{n_i}|\theta\rangle$$

TDA secular equation

The 1-boson eigenvalue equation is equivalent to

$$1 - 2g\sum_{i=1}^{k}\frac{\frac{1}{2}v_i - \frac{1}{4}\Omega_i}{2\varepsilon_i - E_1} = 0,$$

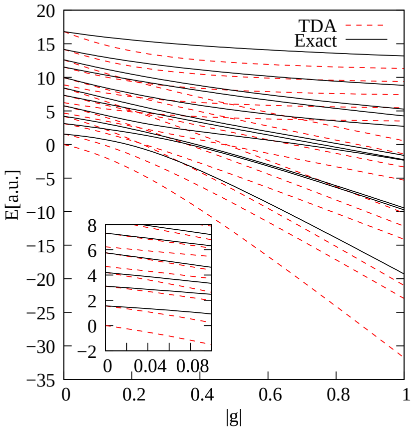


How well does TDA perform?

An example: 4 pairs in 3 levels

i	Ω_i	ε_i
1	6	0.000
2	4	0.780
3	10	2.102

- $dim_{TDA} = 15 > 10 = dim$
- weak interaction: extra states, forbidden by the Pauli principle
- strong interaction: correct clustering, but overbound.
- conclusion: how can we reintroduce the Pauli principle?



A deformed algebra

- We can construct the following deformed algebra

$$[S_i^0, S_i^\dagger] = S_i^\dagger, \quad [S_i^0, S_i] = -S_i, \quad [S_i^\dagger, S_i] = (\xi 2 S_i^0 + (\xi - 1)\tfrac{1}{2}\Omega_i)$$

- So, with ξ, we can control the bosonic character of the fermion pair

- The deformed pairing Hamiltonian *remains* exactly solvable with the

deformed Richardson Gaudin equations

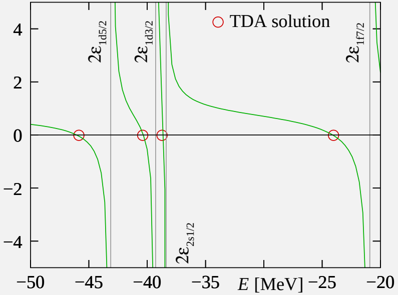
$$1 - g \sum_{i=1}^{k} \frac{\xi v_i - \frac{1}{2}\Omega_i}{2\varepsilon_i - E_\alpha} - 2g\xi \sum_{\beta \neq \alpha}^{N} \frac{1}{E_\beta - E_\alpha} = 0 \qquad (\forall \alpha = 1 \dots N)$$

- $\xi = 1$: we obtain the standard RG equations
- $\xi = 0$: we obtain N seniority-free copies of the TDA equation

From TDA to RG: the technical case

- The Pauli principle can now *adiabatically* be switched on to obtain the solutions of the full RG equations.

starting from TDA



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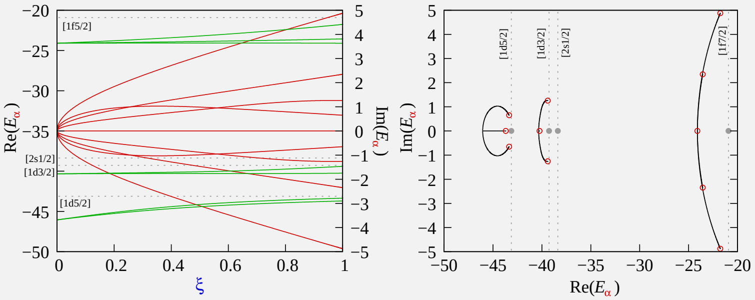
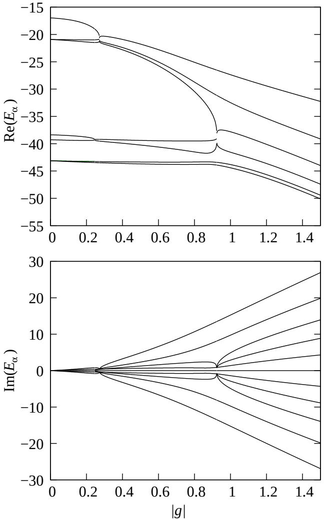


From TDA to RG: the physics case

- We can reconstruct every exact eigenstate from a given TDA solution
- [In reverse, not every TDA prediction leads to an eigen state]
- At the critical points, we see a change of TDA nature. For the ground state of the example, we obtain

$\lvert g \rvert$	0.000	0.245	0.272	0.923
$1d_{5/2}$	3	3	3	11
$1d_{3/2}$	2	3	3	0
$2s_{1/2}$	1	0	0	0
$1f_{7/2}$	4	4	5	0
$2p_{3/2}$	1	1	0	0
$1f_{5/2}$	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$



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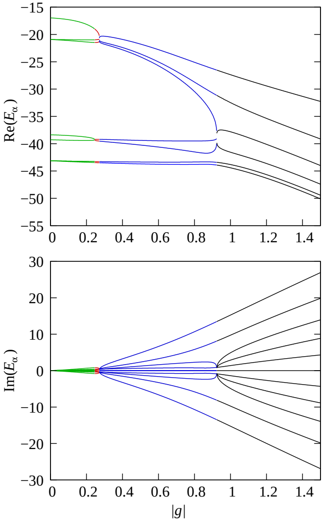

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Remarks and Conclusions

- The pairing model lives in a large Hilbert space
- The reduced BCS Hamiltonian can be diagonalied by means of a Bethe Ansatz wavefunction
- A deformed quasi-spin algebra enables us to obtain the RG variables, starting from the TDA predictions
- The critical points mark the regions with different TDA nature.
- A similar method can be used to solve higher order algebras ($SO(5)$ isoscalar pairing, $SO(8)$ isovector ...)
- The Bethe Ansatz wavefunctions may serve as a good basis for Hamiltonians in too large Hilbert spaces (like Cd, Sn region)

thanks

- Stefan Rombouts (CSIC, Madrid)
- Piet Van Isacker (GANIL, Caen)
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